1 Notes on Fourier series of periodic functions

1.1 Background

Any temporal function can be represented by a multiplicity of basis sets. When the function is assumed to exist for all of time, a not unreasonable approximation for real signals in the steady state, the optimal representation is in the frequency domain. Here we express function $V(t)$ in terms of a continuous expansion in sines and cosines, which are most conveniently written in their complex forms, *i.e.*, sin$x$ $= (e^{ix} - e^{-ix})/(2i)$ and cos$x$ $= (e^{ix} + e^{-ix})/2$. Then

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; \tilde{V}(\omega) \; e^{i\omega t} \tag{1.1}$$

where $\tilde{V}(\omega)$ is a complex function that sets the contribution of different frequencies, ω. The inverse transform is

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt \; V(t) \; e^{-i\omega t}. \tag{1.2}$$

When $V(t)$ is periodic with period T, so that $V(t+T) = V(t)$, we have

$$\int_{-\infty}^{\infty} dt \; V(t) \; e^{-i\omega t} \;\; = \;\; \int_{-\infty}^{\infty} dt \; V(t) \; e^{-i\omega (t+T)} \tag{1.3}$$

so that

$$e^{-i\omega T} \; = \; 1 \tag{1.4}$$

and ω can only take on discrete values, *i.e.*,

$$\omega \; = \; 0, \; \pm\frac{2\pi}{T}, \; \pm\frac{4\pi}{T}, \; \pm\frac{6\pi}{T}, \; ... \; . \tag{1.5}$$

We define $\omega_o = 2\pi/T$ so that

$$\omega \; = \; 0, \; \pm\omega_o, \; \pm 2\omega_o, \; \pm 3\omega_o, \; ... \; . \tag{1.6}$$

and write the Fourier expansion in a discrete form, *i.e.*,

$$V(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k \; e^{ik\omega_o t}. \tag{1.7}$$

The $\tilde{c}_k$ are complex numbers that weight each of the harmonics ω_o. They are defined by

$$\tilde{c}_k = \frac{1}{T} \int_{-T/2}^{+T/2} dt\, V(t)\, e^{-ik\omega_o t}. \tag{1.8}$$

Since we need to end up with sines and cosines, the constant are constrained so that $\tilde{c}_k = \tilde{c}_{-k}$ for $\tilde{c}_k$ real and $\tilde{c}_k = -\tilde{c}_{-k}$ for $\tilde{c}_k$ imaginary.

1.2 Sine waves

This is a trivial case. We have:

$$\begin{aligned} \tilde{c}_0 &= 0 \\ \tilde{c}_{\pm 1} &= \pm\frac{1}{2i} \\ \tilde{c}_{\pm k;\, k\geq 2} &= 0 \end{aligned} \tag{1.9}$$

1.3 Square waves

Here $V(-T/2 < t < 0)$ = -1 and $V(0 < t < T/2) = +1$. We have:

$$\begin{aligned} \tilde{c}_k &= \frac{1}{T}\left[\int_{-T/2}^{0} dt\, (-1)\, e^{-ik\omega_o t} + \int_{0}^{+T/2} dt\, (+1)\, e^{-ik\omega_o t}\right] \\ &= \frac{1}{-ik\omega_o T}\left[- e^{-ik\omega_o t}\big|_{-T/2}^{0} + e^{-ik\omega_o t}\big|_{0}^{T/2}\right] \\ &= \frac{1}{-ik\omega_o T}\left[-1 + e^{+ik\omega_o T/2} + e^{-ik\omega_o T/2} - 1\right] \\ &= \frac{1}{ik\omega_o T}\left[2 - 2\, cos\,(k\omega_o T/2)\right]. \end{aligned} \tag{1.10}$$

We recall that $\omega_o T = 2\pi$, so that

$$\begin{aligned} \tilde{c}_k &= \left(\frac{4}{\pi}\right)\left(\frac{1}{2i}\right)\left[\frac{1 - cos\,(\pi k)}{2k}\right] \\ &= \left(\frac{4}{\pi}\right)\left(\frac{1}{2i}\right)\left(..., 0, -\frac{1}{5}, 0, -\frac{1}{3}, 0, -1, 0, +1, 0, +\frac{1}{3}, 0, +\frac{1}{5}, 0, ...\right) \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} V(t) &= \frac{4}{\pi}\left(\frac{... - \frac{1}{5}e^{-i5\omega_o t} - \frac{1}{3}e^{-i3\omega_o t} - e^{-i\omega_o t} + e^{i\omega_o t} + \frac{1}{3}e^{i3\omega_o t} + \frac{1}{5}e^{i5\omega_o t} + ...}{2i}\right) \\ &= \frac{4}{\pi}\left[sin(\omega_o t) + \frac{1}{3}sin(3\omega_o t) + \frac{1}{5}sin(5\omega_o t) + ...\right]. \end{aligned} \tag{1.12}$$

The result is that the square wave is constructed from a weakly converging set of odd harmonics.

Of potential interest, the "smoother" the function the faster the series for $\tilde{c}_k$ converges, *i.e.*, $\tilde{c}_k$ = constant for a period series of delta functions, *i.e.*, the so-called comb function, $\tilde{c}_k \propto 1/k$ for a square wave as derived above, $\tilde{c}_k \propto 1/k^2$ for a triangular wave, *etc.*